

Generalised differences and a class of multiplier operators in Fourier analysis

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Abstract

Any zeros in the multiplier of an operator impose a condition on a function for it to be in the range of the operator. But if each function in a certain family \mathcal{F} of functions satisfies such a condition, when is \mathcal{F} the range of the operator? Let $\alpha, \beta \in \mathbb{Z}$, for $g \in L^2([0, 2\pi])$ let \hat{g} be the sequence of Fourier coefficients of g , and let D denote differentiation. We consider the operator $D^2 - i(\alpha + \beta)D - \alpha\beta$ on the second order Sobolev space of $L^2([0, 2\pi])$. The multiplier of this operator is $-(n - \alpha)(n - \beta)$, so that $\hat{g}(\alpha) = \hat{g}(\beta) = 0$ for any function g in the range of the operator. Let δ_x denote the Dirac measure at x , and let $*$ denote convolution. If $b \in [0, 2\pi]$ let λ_b be the measure

$$\frac{1}{2} \left[e^{ib(\frac{\alpha-\beta}{2})} + e^{-ib(\frac{\alpha-\beta}{2})} \right] \delta_0 - \frac{1}{2} \left[e^{ib(\frac{\alpha+\beta}{2})} \delta_b + e^{-ib(\frac{\alpha+\beta}{2})} \delta_{-b} \right].$$

A function of the form $\lambda_b * f$ is called a *generalised difference*, and we let \mathcal{F} be the family of functions h such that h is some finite sum of generalised differences. It is shown that \mathcal{F} is a closed subspace of $L^2(\mathbb{T})$ that equals the range of $D^2 - i(\alpha + \beta)D - \alpha\beta$, and that every function in \mathcal{F} is a sum of five generalised differences. The methods use partitions of $[0, \pi/2]$ and estimates of integrals in Euclidean space. There are applications to the automatic continuity of linear forms.

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1 Introduction

The circle group $\{z : |z| = 1\}$ is denoted by \mathbb{T} . The mapping $x \mapsto e^{ix}$ from $[0, 2\pi)$ to \mathbb{T} means that we can and will identify \mathbb{T} with $[0, 2\pi)$ or $[0, 2\pi]$ in the usual way, and both settings will be used (see the comments in [12, page 1034]). The group operation $+$ on $[0, 2\pi)$ is written additively and is the usual addition if $0 \leq x + y < 2\pi$, and is $x + y - 2\pi$ if $2\pi \leq x + y$. The space $L^2(\mathbb{T})$ is the Hilbert space of square integrable complex functions on \mathbb{T} , and $M(\mathbb{T})$ denotes the set of bounded, regular, complex Borel measures on \mathbb{T} . Let \mathbb{Z} denote the set of integers, and let \mathbb{N} denote the set of positive integers. The convolution operation in $M(\mathbb{T})$ is denoted by $*$. Thus, if $\mu \in M(\mathbb{T})$ and $n \in \mathbb{N}$, μ^n denotes $\mu * \mu * \cdots * \mu$, where μ appears n times. Considering $n \in \mathbb{Z}$, $f \in L^2(\mathbb{T})$ and $\mu \in M(\mathbb{T})$, we define the Fourier coefficients $\widehat{f}(n)$ and $\widehat{\mu}(n)$ by

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt \text{ and } \widehat{\mu}(n) = \int_0^{2\pi} e^{-int} d\mu(t).$$

Note that for $\mu, \nu \in M(\mathbb{T})$, $(\mu * \nu)^\wedge = \widehat{\mu} \widehat{\nu}$. Letting δ_x denote the Dirac measure at x , we see that $\widehat{\delta_x}(n) = e^{-inx}$ for $x \in [0, 2\pi]$, and $\widehat{\delta_z}(n) = z^{-n}$ for $z \in \mathbb{T}$. For $s = 0, 1, 2, \dots$ the Sobolev space $W^s(\mathbb{T})$ is defined by

$$W^s(\mathbb{T}) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \sum_{n=-\infty}^{\infty} |n|^{2s} |\widehat{f}(n)|^2 < \infty \right\}.$$

The differential operator D maps $W^1(\mathbb{T})$ into $L^2(\mathbb{T})$ and can be defined by the property

$$D(f)^\wedge(n) = in \widehat{f}(n), \text{ for all } n \in \mathbb{Z}.$$

We say that D is a *multiplier operator on $W^1(\mathbb{T})$ with multiplier in* . Note that this means that if $f \in W^1(\mathbb{T})$, $D(f)^\wedge(0) = 0$. Conversely, it is easy to see that if $g \in L^2(\mathbb{T})$ and $\widehat{g}(0) = 0$, then $g = D(f)$ for some $f \in W^1(\mathbb{T})$. Similarly, D^s is a multiplier operator from $W^s(\mathbb{T})$ into $L^2(\mathbb{T})$ with multiplier $(in)^s$. Let $\alpha, \beta \in \mathbb{Z}$ and $s \in \mathbb{N}$ be given, and let I denote the identity operator. In this paper, and for $s \in \mathbb{N}$, we are concerned with operators $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ that map $W^{2s}(\mathbb{T})$ into $L^2(\mathbb{T})$. These are multiplier operators with multipliers $(-1)^s(n - \alpha)^s(n - \beta)^s$. That is, for all $f \in W^s(\mathbb{T})$,

$$[(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s(f)]^\wedge(n) = (-1)^s(n - \alpha)^s(n - \beta)^s \widehat{f}(n),$$

for all $n \in \mathbb{Z}$. Note that the zeros of the multiplier of $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ are α and β . The operator $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ acting on $f \in W^{2s}(\mathbb{T})$ eliminates the frequencies α and β from f .

A particular case of the above is when $\alpha = -\beta$. Then the operator $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ becomes $(D^2 + \alpha^2 I)^s$, and it has the multiplier $-n^2 + \alpha^2$, which has zeros at $-\alpha$ and α . In order to give a feeling for the ideas in this paper, here is a statement of a special case of what is perhaps the main result. *Let $\alpha \in \mathbb{Z}$. Then the following conditions (i), (ii) and (iii) are equivalent for a function $f \in L^2(\mathbb{T})$.*

(i) $\widehat{f}(-\alpha) = \widehat{f}(\alpha) = 0$.

(ii) f is a sum of five functions, for each one h of which there are $z \in \mathbb{T}$ and $g \in L^2(\mathbb{T})$ such that

$$h = (z^\alpha + z^{-\alpha})g - (\delta_z + \delta_{z^{-1}}) * g. \quad (1.1)$$

(iii) There is $g \in W^2(\mathbb{T})$ such that $(D^2 + \alpha^2 I)(g) = f$.

Now a function of the form $g - \delta_z * g$ might be called a first order difference, and one of the form $(\delta_1 - \delta_z)^s * g$ might be called a difference of order s (see [8, 9]). So, a function as in (1.1) we call a type of *generalised difference*. The form of generalised differences is derived from the factors of the multiplier of the operator, rather than by replacing the elements D and D^2 in the operator by first and second order differences.

The present ideas in proving a result of the above type involve looking at the structure of partitions of $[0, \pi/2]$ associated with the zeros in $[0, \pi/2]$ of the functions $\sin((n - \alpha)x)$ and $\sin((n - \beta)x)$, and estimating integrals in \mathbb{R}^m over sets that are Cartesian products of sets in a refined partition.

The results here are related to work of Meisters and Schmidt [5], where they proved the following result. *The following conditions (i) and (ii) are equivalent for a function $f \in L^2(\mathbb{T})$.*

(i) $\widehat{f}(0) = 0$.

(ii) f is a sum of three functions, for each one h of which there are $z \in \mathbb{T}$ and $g \in L^2(\mathbb{T})$ such that $h = g - \delta_z * g$.

They deduced from this that if T is a linear form on $L^2(\mathbb{T})$ such that $T(f) = T(\delta_z * f)$ for all $z \in \mathbb{T}$ and $f \in L^2(\mathbb{T})$, then T is continuous. That is, any translation invariant linear form on $L^2(\mathbb{T})$ is automatically continuous, and this was proved more generally for compact, connected abelian groups. Further work relating to the original results of Meisters and Schmidt can be found, for example, in [1, 4, 6, 7, 8, 9] where there are further references. One way to think of the ideas in [5] is that they are

concerned with the range of the differential operator D whose multiplier is a linear polynomial. On the other hand, the present work is concerned with differential operators whose multipliers are quadratic in nature. The work here goes back to [5], but takes some of the ideas there in a different direction from the mainstream of later work. There are also applications to automatic continuity of linear forms. A suitable general reference on classical Fourier series is [2], and for abstract harmonic analysis on groups see [3, 12].

2 Background and statement of the main result

There will be cause to consider series of the form $\sum_{j=-\infty}^{\infty} a_j/b_j$, where $a_j, b_j \geq 0$ for all j . In such a case, we write $\sum_{j=-\infty}^{\infty} a_j/b_j = \infty$ if there is a term of the form $a_j/0$ with $a_j > 0$. If there are any terms of the form $0/0$ we either neglect them in the sum or make them equal to 0. We write respectively $\sum_{j=-\infty}^{\infty} a_j/b_j < \infty$ or $\sum_{j=-\infty}^{\infty} a_j/b_j = \infty$ if the series $\sum_{j=-\infty, a_j > 0}^{\infty} a_j/b_j$ converges or diverges in the usual sense. The following result is due essentially to Meisters and Schmidt [5, page 413].

Theorem 2.1 *Let $f \in L^2([0, 2\pi])$ and let $\mu_1, \mu_2, \dots, \mu_r \in M([0, 2\pi])$. Then the following conditions (i) and (ii) are equivalent.*

(i) *There are $f_1, f_2, \dots, f_r \in L^2([0, 2\pi])$ such that $f = \sum_{j=1}^r \mu_j * f_j$.*

$$(ii) \quad \sum_{n=-\infty}^{\infty} \frac{|\widehat{f}(n)|^2}{\sum_{j=1}^r |\widehat{u}_j(n)|^2} < \infty.$$

Proof. This is essentially proved in [5, pages 411-412], but see also [8, pages 77-88] and [9, page 23]. A more accessible proof for the present context is on the world wide web [10]. \square

In [5], the measures in Theorem 2.1 were taken to be of the form $\mu_b = \delta_0 - \delta_b$, in which case $\mu_b([0, 2\pi]) = 0$ and $\widehat{\mu}_b(n) = 1 - e^{-ibn}$. In the context here, we let $\alpha, \beta \in \mathbb{Z}$ and we will apply Theorem 2.1 using measures λ_b , $b \in [0, 2\pi)$, where

$$\lambda_b = \frac{1}{2} \left[e^{ib(\frac{\alpha-\beta}{2})} + e^{-ib(\frac{\alpha-\beta}{2})} \right] \delta_0 - \frac{1}{2} \left[e^{ib(\frac{\alpha+\beta}{2})} \delta_b + e^{-ib(\frac{\alpha+\beta}{2})} \delta_{-b} \right]. \quad (2.1)$$

Note the the Fourier transform $\widehat{\lambda}_b$ of λ_b is given for $n \in \mathbb{Z}$ by

$$\widehat{\lambda}_b(n) = \cos \left(\left(\frac{\alpha - \beta}{2} \right) b \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) b \right). \quad (2.2)$$

Consequently,

$$\widehat{\lambda}_b(\alpha) = \widehat{\lambda}_b(\beta) = 0, \text{ for all } b \in [0, 2\pi]. \quad (2.3)$$

So if $b \in [0, 2\pi]$ and $g \in L^2([0, 2\pi])$,

$$\lambda_b * g = \frac{1}{2} \left[e^{ib(\frac{\alpha-\beta}{2})} + e^{-ib(\frac{\alpha-\beta}{2})} \right] g - \frac{1}{2} \left[e^{ib(\frac{\alpha+\beta}{2})} \delta_b + e^{-ib(\frac{\alpha+\beta}{2})} \delta_{-b} \right] * g, \quad (2.4)$$

and we see that if $f \in L^2([0, 2\pi])$ is a function of the form $\lambda_b * g$, then $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$. A function of the form $\lambda_b * g$, as in (2.4) above, is called a *generalised difference* in $L^2([0, 2\pi])$.

The following is a crucial technical result. It is not proved here, but is a consequence of results in subsequent sections.

Lemma 2.2 *Let $\alpha, \beta \in \mathbb{Z}$ and let $s \in \mathbb{N}$ be given. Then there is $M > 0$ such that for all $n \in \mathbb{Z}$ with $n \notin \{\alpha, \beta\}$,*

$$\int_{[0, 2\pi]^{4s+1}} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^{4s+1} \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \leq M.$$

The central result proved here using Lemma 2.2 is the following.

Theorem 2.3 *Let $\alpha, \beta \in \mathbb{Z}$. Then the following conditions (i), (ii), (iii) and (iv) on a function $f \in L^2([0, 2\pi])$ are equivalent.*

(i) $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$.

(ii) *There are $m, s \in \mathbb{N}$, $b_1, b_2, \dots, b_m \in [0, 2\pi]$ and $f_1, f_2, \dots, f_m \in L^2([0, 2\pi])$ such that f is equal to*

$$\sum_{j=1}^m \left[\left(e^{ib_j(\frac{\alpha-\beta}{2})} + e^{-ib_j(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left(e^{ib_j(\frac{\alpha+\beta}{2})} \delta_{b_j} + e^{-ib_j(\frac{\alpha+\beta}{2})} \delta_{-b_j} \right) \right]^s * f_j. \quad (2.5)$$

(iii) *There are $s \in \mathbb{N}$, $b_1, b_2, \dots, b_{4s+1} \in [0, 2\pi]$ and $f_1, f_2, \dots, f_{4s+1} \in L^2([0, 2\pi])$ such that f is equal to*

$$\sum_{j=1}^{4s+1} \left[\left(e^{ib_j(\frac{\alpha-\beta}{2})} + e^{-ib_j(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left(e^{ib_j(\frac{\alpha+\beta}{2})} \delta_{b_j} + e^{-ib_j(\frac{\alpha+\beta}{2})} \delta_{-b_j} \right) \right]^s * f_j. \quad (2.6)$$

(iv) *There are $s \in \mathbb{N}$ and $g \in W^{2s}([0, 2\pi])$ such that*

$$(D^2 - i(\alpha + \beta)D - \alpha\beta I)(g) = f. \quad (2.7)$$

When the equivalent conditions (i), (ii), (iii) and (iv) are satisfied and $s \in \mathbb{N}$ is given, we have that for almost all $(b_1, b_2, \dots, b_{4s+1}) \in [0, 2\pi]^{4s+1}$, there are $f_1, f_2, \dots, f_{4s+1} \in L^2([0, 2\pi])$ such that (2.6) holds. Also, the functions in $L^2([0, 2\pi])$ that can be written in the form (2.6) form a closed vector subspace of $L^2([0, 2\pi])$.

Proof. It is obvious that (iii) implies (ii). Now, (ii) implies (i) since, as noted in (2.3), the Fourier coefficients of any function appearing within the sum in (2.5) vanish at α and β .

In order to prove that (i) implies (iii), let $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$, and let M be the constant as in Lemma 2.2. If we integrate the function that maps $(x_1, x_2, \dots, x_{4s+1}) \in \mathbb{R}^{4s+1}$ into

$$\sum_{\substack{n=-\infty \\ n \neq \alpha, \beta}} \frac{|\widehat{f}(n)|^2}{\sum_{j=1}^{4s+1} \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \quad (2.8)$$

over $[0, 2\pi]^{4s+1}$, and interchange the order of integration and summation we obtain

$$\begin{aligned} & \sum_{\substack{n=-\infty \\ n \neq \alpha, \beta}}^{\infty} \left(\int_{[0, 2\pi]^{4s+1}} \frac{dx_1 dx_2 \cdots dx_{4s+1}}{\sum_{j=1}^{4s+1} \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \right) |\widehat{f}(n)|^2 \\ & \leq M \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \\ & < \infty. \end{aligned}$$

We deduce that for almost all $(x_1, x_2, \dots, x_{4s+1}) \in [0, 2\pi]^{4s+1}$, the sum in (2.8) is finite. Using (2.2), we see that for almost all $(x_1, \dots, x_{4s+1}) \in [0, 2\pi]^{4s+1}$,

$$\sum_{n=-\infty}^{\infty} \frac{|\widehat{f}(n)|^2}{\sum_{j=1}^{4s+1} |\widehat{\lambda}_{x_j}^s(n)|^2}$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{\widehat{f}(n)^2}{\sum_{j=1}^{4s+1} |\widehat{\lambda}_{x_j}(n)|^{2s}} \\
&= \sum_{n=-\infty}^{\infty} \frac{\widehat{f}(n)^2}{\sum_{j=1}^{4s+1} |\widehat{\lambda}_{x_j}(n)|^{2s}} \\
&= \sum_{\substack{n=-\infty \\ n \neq \alpha, \beta}}^{\infty} \frac{|\widehat{f}(n)|^2}{\sum_{j=1}^{4s+1} \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \\
&< \infty.
\end{aligned}$$

It now follows from (2.2), and the equivalence of (i) and (ii) in Theorem 2.1, that for almost all $(b_1, b_2, \dots, b_{4s+1}) \in [0, 2\pi]^{4s+1}$ there are $f_1, f_2, \dots, f_{4s+1} \in L^2([0, 2\pi])$ such that

$$f = \sum_{j=1}^{4s+1} \lambda_{b_j}^s * f_j.$$

We see, using (2.1), that (2.6) and hence (iii) hold.

Now to see that (iv) implies (i), we observe that the multiplier of $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ is $(-1)^s(n - \alpha)^s(n - \beta)^s$. Thus, if $f = (D^2 - i(\alpha + \beta)D - \alpha\beta I)^s(g)$, as in (2.7), $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$.

Now, we prove (i) implies (iv). If $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$, define g in terms of \widehat{g} by putting $\widehat{g}(\alpha) = \widehat{g}(\beta) = 0$ and $\widehat{g}(n) = (-1)^s \widehat{f}(n) / (n - \alpha)^s(n - \beta)^s$, for $n \neq \alpha$ and $n \neq \beta$. Then $\sum_{n=-\infty}^{\infty} |n|^{4s} |\widehat{g}(n)|^2 < \infty$, so $g \in W^{2s}([0, 2\pi])$. Also, as the multiplier of $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ is $(-1)^s(n - \alpha)^s(n - \beta)^s$, we see that $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s(g)^\wedge = \widehat{f}$ and so $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s(g) = f$.

Finally, that the functions expressible as in (2.6) form a closed subspace follows from the fact that such functions are characterised by (i). \square

3 Partitioning, zeros and inequalities

In this section we make observations and obtain results for the later proving of Lemma 2.2.

Lemma 3.1 *Let $m, s \in \mathbb{N}$ and $\alpha, \beta, n \in \mathbb{Z}$. Then,*

$$\begin{aligned} & \int_{[0, 2\pi]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \\ &= 2^{2m-2s} \int_{[0, \pi/2]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \sin^{2s}((n - \alpha)x_j) \sin^{2s}((n - \beta)x_j)}, \end{aligned} \quad (3.1)$$

but note that both integrals may be infinite.

Proof. The use of a familiar trigonometric formula and a simple substitution gives

$$\begin{aligned} & \int_{[0, 2\pi]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \\ &= 2^{m-2s} \int_{[0, \pi]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \sin^{2s}((n - \alpha)x_j) \sin^{2s}((n - \beta)x_j)}. \end{aligned} \quad (3.2)$$

Note that for $x \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, $|\sin(\ell x)| = |\sin(\ell(\pi - x))|$. Also, note that $[0, \pi]^m$ is the disjoint union of the 2^m sets $\prod_{t=1}^m J_t$ where, for each t , J_t is either $[0, \pi/2)$ or $[\pi/2, \pi)$. Using substitutions for $\pi - x_j$, as needed in (3.2), we see that (3.1) follows from (3.2). \square

We are aiming to estimate the integral in Lemma 3.1. Motivated by (3.1), we consider the zeros in $[0, \pi/2]$ of $\sin((n - \alpha)x)$ and $\sin((n - \beta)x)$. Some preliminary notions are needed.

Definitions. If J is an interval we denote its length by $\lambda(J)$. Let $[a, b]$ be a closed interval with $\lambda([a, b]) > 0$. A family $\{J_0, J_1, \dots, J_{r-1}\}$ of closed intervals having non-empty interiors is a *partition* of $[a, b]$ if $\cup_{j=0}^{r-1} J_j = [a, b]$ and any two intervals in the family have at most a single point in common. In such a case, the intervals may be arranged so that the right endpoint of J_{j-1} is the left endpoint of J_j for all $j = 1, 2, \dots, r - 1$. Note that in the sense used here, the sets in a partition are not pairwise disjoint.

Lemma 3.2 *Let $[a, b]$ be a closed interval with $\lambda([a, b]) > 0$. Let R_0, R_1, \dots, R_{r-1} be closed intervals in a partition \mathcal{P}_1 of $[a, b]$. Let S_0, S_1, \dots, S_{s-1} be*

closed intervals in a partition \mathcal{P}_2 of $[a, b]$. Let

$$\mathcal{A} = \{(j, k) : 0 \leq j \leq r-1, 0 \leq k \leq s-1 \text{ and } \lambda(R_j \cap S_k) > 0\},$$

and put

$$\mathcal{P} = \{R_j \cap S_k : (j, k) \in \mathcal{A}\}. \quad (3.3)$$

Then, \mathcal{P} is a partition of $[a, b]$ and the number of intervals in \mathcal{P} is at most

$$r + s - 1. \quad (3.4)$$

Proof. It is clear that $\mathcal{P} = \{R_j \cap S_k : (j, k) \in \mathcal{A}\}$ as in (3.3) is a partition of $[a, b]$. Given any two partitions $\mathcal{P}_1, \mathcal{P}_2$ as in the Lemma, we will denote the partition given as in (3.3) by $\mathcal{P}(\mathcal{P}_1, \mathcal{P}_2)$.

We proceed by induction on r . If $r = 1$, $\mathcal{P}_1 = \{R_0\} = \{[a, b]\}$, so that $\mathcal{P}(\mathcal{P}_1, \mathcal{P}_2) = \mathcal{P}_2$ and $\mathcal{P}(\mathcal{P}_1, \mathcal{P}_2)$ has s elements. In this case, $s = r + s - 1$, and we see that when $r = 1$, (3.4) holds for all $s \in \mathbb{N}$. Similarly, it is easy to see that when $r = 2$, (3.4) holds for all $s \in \mathbb{N}$.

So, let $r \in \mathbb{N}$ with $r \geq 3$ be such that (3.4) holds for all partitions \mathcal{P}_1 having r intervals and for all partitions \mathcal{P}_2 having any number of intervals. Let $\mathcal{P}_3 = \{Q_0, Q_1, \dots, Q_r\}$ be a partition of $[a, b]$ into $r + 1$ intervals and let $\mathcal{P}_2 = \{S_0, S_1, \dots, S_{s-1}\}$ be a partition of $[a, b]$ into s intervals. Consider the partition $\mathcal{P}_1 = \{Q_0 \cup Q_1, Q_2, \dots, Q_r\}$, which has r intervals. By the inductive hypothesis, $\mathcal{P}(\mathcal{P}_1, \mathcal{P}_2)$ has at most $r + s - 1$ intervals. Let ξ be the right endpoint of Q_0 , which is also the left endpoint of Q_1 . Now, in passing from \mathcal{P}_1 to \mathcal{P}_3 , we do so by using ξ to divide the single interval $Q_0 \cup Q_1$ into the two intervals Q_0 and Q_1 . If ξ is not an endpoint of any interval in \mathcal{P}_2 , this will divide one interval in $\mathcal{P}(\mathcal{P}_1, \mathcal{P}_2)$ into two subintervals belonging to $\mathcal{P}(\mathcal{P}_2, \mathcal{P}_3)$. If ξ is an endpoint of some interval in \mathcal{P}_2 , this division does not increase the number of intervals in $\mathcal{P}(\mathcal{P}_2, \mathcal{P}_3)$ in going from \mathcal{P}_1 to \mathcal{P}_3 . In either case, we see that $\mathcal{P}(\mathcal{P}_2, \mathcal{P}_3)$ has at most

$$r + s - 1 + 1 = r + s = (r + 1) + s - 1$$

intervals, showing that (3.4) holds with $r + 1$ in place of r . The result follows by induction. \square

Definition. Let $[a, b]$ be a closed interval of positive length. Let $\mathcal{P}_1 = \{R_0, R_1, \dots, R_{r-1}\}$ and $\mathcal{P}_2 = \{S_0, S_1, \dots, S_{s-1}\}$ be two partitions of $[a, b]$. Let \mathcal{P} be the partition of $[a, b]$ as given by (3.3) in Lemma 3.2. Then \mathcal{P} is called the *refinement* of the partitions \mathcal{P}_1 and \mathcal{P}_2 .

Now, let $n, \gamma \in \mathbb{Z}$ with $n \neq \gamma$ be given. We construct an associated partition $\mathcal{P}(\gamma)$ of $[0, \pi/2]$, as follows.

(i) When $|n - \gamma|$ is even we define $(|n - \gamma| + 2)/2$ closed subintervals $Q_0, Q_1, \dots, Q_{(|n - \gamma| + 2)/2}$ of $[0, \pi/2]$ by putting

$$Q_0 = \left[0, \frac{\pi}{2|n - \gamma|}\right], Q_{(|n - \gamma| + 2)/2} = \left[\frac{\pi(|n - \gamma| - 1)}{2|n - \gamma|}, \frac{\pi}{2}\right], \text{ and} \\ Q_j = \left[\frac{\pi(j - 1/2)}{|n - \gamma|}, \frac{\pi(j + 1/2)}{|n - \gamma|}\right], \quad (3.5)$$

for $j = 1, 2, \dots, (|n - \gamma| - 2)/2$.

(ii) When $|n - \gamma|$ is odd we define $(|n - \gamma| + 1)/2$ closed subintervals $Q_0, Q_1, \dots, Q_{(|n - \gamma| + 1)/2}$ of $[0, \pi/2]$ by putting

$$Q_0 = \left[0, \frac{\pi}{2|n - \gamma|}\right] \text{ and } Q_j = \left[\frac{\pi(j - 1/2)}{|n - \gamma|}, \frac{\pi(j + 1/2)}{|n - \gamma|}\right], \quad (3.6)$$

the latter for $j = 1, 2, \dots, (|n - \gamma| - 1)/2$.

Put $\theta(r) = (r + 2)/2$ if r is even, and $\theta(r) = (r + 1)/2$ if r is odd. Then, with n and γ as given, (3.5) and (3.6) above define $\theta(|n - \gamma|)$ closed subintervals $Q_0, Q_1, \dots, Q_{\theta(|n - \gamma|) - 1}$ of $[0, \pi/2]$. We put

$$\mathcal{P}(\gamma) = \{Q_0, Q_1, Q_2, \dots, Q_{\theta(|n - \gamma|) - 1}\}. \quad (3.7)$$

Note that $\mathcal{P}(\gamma)$ depends only upon $|n - \gamma|$ so, strictly speaking, $\mathcal{P}(\gamma)$ depends also on n as well as γ . The significance of $\mathcal{P}(\gamma)$ lies in its relationship to the zeros of $\sin(n - \gamma)x$ in $[0, \pi/2]$. There are $\theta(|n - \gamma|)$ zeros $c_1, c_2, \dots, c_{\theta(|n - \gamma|)}$ of $\sin(n - \gamma)x$ in $[0, \pi/2]$ given by

$$c_j = \frac{\pi j}{|n - \gamma|}, \text{ for } j = 0, 1, 2, \dots, \theta(|n - \gamma|) - 1. \quad (3.8)$$

Now, we see from (3.5) and (3.6) that c_0 is the left endpoint of Q_0 , when $|n - \gamma|$ is even c_j is the midpoint of Q_j for $j = 1, 2, \dots, \theta(|n - \gamma|) - 2$ and $c_{\theta(|n - \gamma|) - 1} = \pi/2$ is the right endpoint of $Q_{\theta(|n - \gamma|) - 1}$ and, when $|n - \gamma|$ is odd c_j is the midpoint of Q_j for $j = 1, 2, \dots, \theta(|n - \gamma|) - 1$.

Lemma 3.3 *Let $\alpha, \beta, n \in \mathbb{Z}$ be such that $n \neq \alpha$ and $n \neq \beta$. Let $\mathcal{P}(\alpha, \beta)$ be the partition of $[0, \pi/2]$ that is the refinement of $\mathcal{P}(\alpha)$ and $\mathcal{P}(\beta)$, as given by (3.3). Then the number of intervals in $\mathcal{P}(\alpha, \beta)$ is bounded above by*

$$2 \max\{|n - \alpha|, |n - \beta|\}. \quad (3.9)$$

Also, if $J \in \mathcal{P}(\alpha, \beta)$,

$$0 < \lambda(J) \leq \min \left\{ \frac{\pi}{|n - \alpha|}, \frac{\pi}{|n - \beta|} \right\}. \quad (3.10)$$

Proof. The partition $\mathcal{P}(\alpha)$ has $\theta(|n - \alpha|)$ intervals, while $\mathcal{P}(\beta)$ has $\theta(|n - \beta|)$ intervals. So, we see from (3.4) that $\mathcal{P}(\alpha, \beta)$ has at most $\theta(|n - \alpha|) + \theta(|n - \beta|) - 1$ intervals. However, $\theta(r) \leq (r + 2)/2$ for all $r \in \mathbb{N}$, so an upper bound for the number of intervals in $\mathcal{P}(\alpha, \beta)$ is

$$\frac{1}{2}(|n - \alpha| + |n - \beta|) + 1 \leq \max\{|n - \alpha|, |n - \beta|\} + 1 \leq 2 \max\{|n - \alpha|, |n - \beta|\}.$$

Finally, if $J \in \mathcal{P}(\alpha, \beta)$, $J = R \cap S$ for some $R \in \mathcal{P}(\alpha)$ and $S \in \mathcal{P}(\beta)$. Then, $\lambda(R) \leq \pi/|n - \alpha|$ and $\lambda(S) \leq \pi/|n - \beta|$, and so (3.10) follows. \square

Figure 1 illustrates Lemma 3.3 in the case $\alpha = 1$, $\beta = -1$ and $n = 9$.

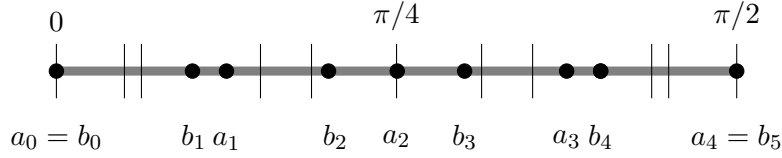


Figure 1. The figure illustrates the case $n = 9$, $\alpha = 1$, $\beta = -1$. We have $n - \alpha = 8$, $n - \beta = 10$. The zeros of $\sin 8x$ and $\sin 10x$ in $[0, \pi/2]$ are denoted respectively by a_j and b_j . The vertical lines in the figure illustrate the intervals in the refinement $\mathcal{P}(-1, 1)$ of $\mathcal{P}(-1)$ and $\mathcal{P}(1)$. Note that four of the ten intervals in $\mathcal{P}(-1, 1)$ contain no zeros of $\sin 8x \sin 10x$.

Now, if $x \in \mathbb{R}$, let $d_{\mathbb{Z}}(x)$ denote the distance from x to a nearest integer. For later use, note the fact that $d_{\mathbb{Z}}(x) = |x|$ if and only if $-1/2 \leq x \leq 1/2$.

Lemma 3.4 *Let $n, \alpha, \beta \in \mathbb{Z}$ with $n \neq \alpha$, $n \neq \beta$. Let the partitions $\mathcal{P}(\alpha)$ and $\mathcal{P}(\beta)$ be given as in (3.7), and we write*

$$\mathcal{P}(\alpha) = \{R_0, R_1, \dots, R_{\theta(|n - \alpha|) - 1}\} \text{ and } \mathcal{P}(\beta) = \{S_0, S_1, \dots, S_{\theta(|n - \beta|) - 1}\}.$$

Let $R_j \cap S_k$ be an element of the refinement $\mathcal{P}(\alpha, \beta)$ of $\mathcal{P}(\alpha)$ and $\mathcal{P}(\beta)$. Then, if $x \in R_j \cap S_k$, we have

$$\begin{aligned} & \sin^2((n - \alpha)x) \sin^2((n - \beta)x) \\ & \geq \frac{2^4(n - \alpha)^2(n - \beta)^2}{\pi^4} \left(x - \frac{j\pi}{|n - \alpha|} \right)^2 \left(x - \frac{k\pi}{|n - \beta|} \right)^2. \end{aligned} \quad (3.11)$$

Proof. We will use the fact that for all $x \in \mathbb{R}$, $|\sin \pi x| \geq 2d_{\mathbb{Z}}(x)$ [9, page 89, for example]. Given that $x \in R_j$, we see from (3.5) and (3.6) that

$$\left| (n - \alpha) \left(\frac{x}{\pi} - \frac{j}{|n - \alpha|} \right) \right| \leq \frac{1}{2}. \quad (3.12)$$

Using (3.12) we now have, for all x in R_j ,

$$\begin{aligned} |\sin((n - \alpha)x)| &= |\sin(|n - \alpha|x - j\pi)| \\ &= \left| \sin \left(\pi(n - \alpha) \left(\frac{x}{\pi} - \frac{j}{|n - \alpha|} \right) \right) \right| \\ &\geq 2d_{\mathbb{Z}} \left((n - \alpha) \left(\frac{x}{\pi} - \frac{j}{|n - \alpha|} \right) \right) \\ &= 2|n - \alpha| \left| \frac{x}{\pi} - \frac{j}{|n - \alpha|} \right| \\ &= \frac{2}{\pi} |n - \alpha| \left| x - \frac{j\pi}{|n - \alpha|} \right|. \end{aligned} \quad (3.13)$$

Using a corresponding argument, we see also that for all $x \in S_k$,

$$|\sin((n - \beta)x)| \geq \frac{2}{\pi} |n - \beta| \left| x - \frac{k\pi}{|n - \beta|} \right|. \quad (3.14)$$

Conclusion (3.11) now follows from (3.13) and (3.14). \square

4 Integral estimates in \mathbb{R}^m

In this section we develop estimates for some integrals in \mathbb{R}^m , and an inequality between quadratics, with a view to proving Lemma 2.2.

Lemma 4.1 *Let $s, m \in \mathbb{N}$ with $m \geq 4s + 1$. Then, there is a number $M > 0$, depending upon s and m only, such that for all $b_1, b_2, \dots, b_m > 0$ and for all $(a_1, a_2, \dots, a_m) \in \prod_{t=1}^m [-b_t, b_t]$,*

$$\int_{\prod_{t=1}^m [-b_t, b_t]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t^2 - a_t^2)^{2s}} \leq M \left(\max \{b_1, b_2, \dots, b_m\} \right)^{m-4s}.$$

PROOF. Clearly, we may assume that $0 \leq a_t \leq b_t$ for all $t = 1, 2, \dots, m$. Now, we have

$$\begin{aligned}
& \int_{\prod_{t=1}^m [-b_t, b_t]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t^2 - a_t^2)^{2s}} \\
&= 2^m \int_{\prod_{t=1}^m [0, b_t]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t^2 - a_t^2)^{2s}} \\
&= 2^m \int_{\prod_{t=1}^m [0, b_t]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t - a_t)^{2s} (u_t + a_t)^{2s}} \\
&= 2^m \int_{\prod_{j=1}^m [-a_t, b_t - a_t]} \frac{dv_1 dv_2 \dots dv_m}{\sum_{t=1}^m v_t^{2s} (v_t + 2a_t)^{2s}}, \tag{4.1}
\end{aligned}$$

on putting $v_t = u_t - a_t$. Now observe that if $v_t \geq 0$ then $v_t + 2a_t \geq 0$, and that if $-a_t \leq v_t \leq 0$ then $v_t + 2a_t \geq a_t \geq |v_t|$. Also, there is $C_m > 0$ such that for all $(v_1, v_2, \dots, v_m) \in \mathbb{R}^m$,

$$\sum_{j=1}^m v_j^{4s} \geq C_m \left(\sum_{j=1}^m v_j^2 \right)^{2s}. \tag{4.2}$$

If $r > 0$, we denote the closed sphere $\{x : x \in \mathbb{R}^m \text{ and } |x| \leq r\}$ by $S(0, r)$. Also, we put $b = \max\{b_1, b_2, \dots, b_m\}$. Using the preceding observations and (4.1) we have

$$\begin{aligned}
& \int_{\prod_{t=1}^m [-b_t, b_t]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t^2 - a_t^2)^{2s}} \\
& \leq 2^m \int_{\prod_{j=1}^m [-a_t, b_t - a_t]} \frac{dv_1 dv_2 \dots dv_m}{\sum_{t=1}^m v_t^{4s}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2^m}{C_m} \int_{\prod_{j=1}^m [-a_t, b_t - a_t]} \frac{dv_1 dv_2 \dots dv_m}{\left(\sum_{t=1}^m v_t^2\right)^{2s}}, \text{ using (4.2),} \\
&\leq \frac{2^m}{C_m} \int_{\prod_{j=1}^m [-b_t, b_t]} \frac{dv_1 dv_2 \dots dv_m}{\left(\sum_{t=1}^m v_t^2\right)^{2s}}, \\
&\quad \text{as } [-a_t, b_t - a_t] \subseteq [-b, b]^m, \\
&\leq \frac{2^m}{C_m} \int_{S(0, b\sqrt{m})} \frac{dv_1 dv_2 \dots dv_m}{\left(\sum_{t=1}^m v_t^2\right)^{2s}}, \text{ as } [-b, b]^m \subseteq S(0, b\sqrt{m}), \\
&= \frac{2^{m+1} \pi^{m/2}}{C_m \Gamma(m/2)} \int_0^{b\sqrt{m}} r^{m-4s-1} dr, \text{ by [14, pages 394-395],} \\
&= \frac{2^{m+1} \pi^{m/2} m^{(m-4s)/2} b^{m-4s}}{C_m \Gamma(m/2)(m-4s)}.
\end{aligned}$$

So, the result holds if $M = C_m^{-1} 2^{m+1} \pi^{m/2} m^{(m-4s)/2} (m-4s)^{-1} \Gamma(m/2)^{-1}$. \square

Lemma 4.2 *Let $s, m \in \mathbb{N}$ with $m \geq 4s + 1$, and let numbers $b_{t,k}, c_t$ and d_t be given for all t, k with $t = 1, 2, \dots, m$ and $k = 1, 2$. We assume that*

$$0 \leq b_{t,1} \leq c_t \leq d_t \leq b_{t,2}$$

for all $t = 1, 2, \dots, m$. Then, there is a number $M > 0$, depending upon s and m only and independent of $b_{t,k}, c_t$ and d_t , such that

$$\begin{aligned}
&\int_{\prod_{t=1}^m [b_{t,1}, b_{t,2}]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t - c_t)^{2s} (u_t - d_t)^{2s}} \\
&\leq M \left(\max \left\{ b_{1,2} - b_{1,1}, b_{2,2} - b_{2,1}, \dots, b_{m,2} - b_{m,1} \right\} \right)^{m-4s}.
\end{aligned}$$

Proof. Put, for $t = 1, 2, \dots, m$,

$$\eta_t = \frac{c_t + d_t}{2}, \gamma_t = \frac{d_t - c_t}{2}, v_t = u_t - \eta_t.$$

Note that $\eta_t \in [b_{t,1}, b_{t,2}]$ and $0 \leq \gamma_t \leq (b_{t,2} - b_{t,1})/2$. Using Lemma 4.1 and putting $v_t = u_t - \eta_t$ in the following we have

$$\begin{aligned}
& \int_{\prod_{t=1}^m [b_{t,1}, b_{t,2}]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t - c_t)^{2s} (u_t - d_t)^{2s}} \\
&= \int_{\prod_{t=1}^m [b_{t,1} - \eta_t, b_{t,2} - \eta_t]} \frac{dv_1 dv_2 \dots dv_m}{\sum_{t=1}^m (v_t + \gamma_t)^{2s} (v_t - \gamma_t)^{2s}} \\
&= \int_{\prod_{t=1}^m [-(\eta_t - b_{t,1}), b_{t,2} - \eta_t]} \frac{dv_1 dv_2 \dots dv_m}{\sum_{t=1}^m (v_t^2 - \gamma_t^2)^{2s}} \\
&\leq \int_{\prod_{t=1}^m [-(b_{t,2} - b_{t,1}), b_{t,2} - b_{t,1}]} \frac{dv_1 dv_2 \dots dv_m}{\sum_{t=1}^m (v_t^2 - \gamma_t^2)^{2s}},
\end{aligned}$$

as $[-(\eta_t - b_{t,1}), b_{t,2} - \eta_t] \subseteq [-(b_{t,2} - b_{t,1}), b_{t,2} - b_{t,1}]$.

We now see that Lemma 4.1 applies because $0 \leq \gamma_t \leq b_{t,2} - b_{t,1}$, and so there is a constant $M > 0$, depending only upon s and m , such that

$$\begin{aligned}
& \int_{\prod_{t=1}^m [b_{t,1}, b_{t,2}]} \frac{du_1 du_2 \dots du_m}{\sum_{t=1}^m (u_t - c_t)^{2s} (u_t - d_t)^{2s}} \\
&\leq M \left(\max \left\{ b_{1,2} - b_{1,1}, b_{2,2} - b_{2,1}, \dots, b_{m,2} - b_{m,1} \right\} \right)^{m-4s}.
\end{aligned}$$

□

Lemma 4.3 *Let $c \leq a < b \leq d$. Let f, g be the quadratic functions given by $f(x) = (x - c)(d - x)$, $g(x) = (x - a)(b - x)$. Then, $f(x) \geq g(x) \geq 0$ for all $a \leq x \leq b$.*

Proof. We have $(f - g)(x) = x(c + d - a - b) + ab - cd$. So,

$$(f - g)(a) = (a - c)(d - a) \geq 0 \text{ and } (f - g)(b) = (d - b)(b - c) \geq 0.$$

As $f - g$ is linear and non-negative at a and b , we deduce that $f(x) \geq g(x)$ for all $x \in [a, b]$. □

5 Completion of the proof of Theorem 2.3

Let $s \in \mathbb{N}$ and $n, \alpha, \beta \in \mathbb{Z}$ with $n \neq \alpha$ and $n \neq \beta$. Let $a_0, a_1, \dots, a_{\theta(|n-\alpha|)-1}$ and $b_0, b_1, \dots, b_{\theta(|n-\beta|)-1}$ respectively be the zeros of $\sin(n-\alpha)x$ and $\sin(n-\beta)x$ in $[0, \pi/2]$, as given correspondingly for $\sin(n-\gamma)x$ in (3.8). Let $\mathcal{P}(\alpha) = \{R_0, R_1, \dots, R_{\theta(|n-\alpha|)-1}\}$ and $\mathcal{P}(\beta) = \{S_0, S_1, \dots, S_{\theta(|n-\beta|)-1}\}$ be the partitions as given by (3.7), and recall that \mathcal{A} is the set of all (j, k) such that $\lambda(R_j \cap S_k) > 0$ and that the partition $\mathcal{P}(\alpha, \beta)$ of $[0, \pi/2]$ is the set $\{R_j \cap S_k : (j, k) \in \mathcal{A}\}$, as in Lemma 3.3. We see from (3.1) of Lemma 3.1 and from (3.11) of Lemma 3.4 that for any $m, s \in \mathbb{N}$,

$$\begin{aligned} & \int_{[0, 2\pi]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \left| \cos\left(\left(\frac{\alpha-\beta}{2}\right)x_j\right) - \cos\left(\left(n - \frac{\alpha+\beta}{2}\right)x_j\right) \right|^{2s}} \\ &= 2^{2m-2s} \int_{[0, \pi/2]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \sin^{2s}((n-\alpha)x_j) \sin^{2s}((n-\beta)x_j)} \\ &\leq \frac{2^{2m-6s} \pi^{4s}}{(n-\alpha)^{2s} (n-\beta)^{2s}} \sum_{(j_1, k_1), \dots, (j_m, k_m) \in \mathcal{A}} J((j_1, k_1), \dots, (j_m, k_m)), \quad (5.3) \end{aligned}$$

where

$$J((j_1, k_1), \dots, (j_m, k_m)) = \int_{\prod_{t=1}^m R_{j_t} \cap S_{k_t}} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{t=1}^m (x_t - a_{j_t})^{2s} (x_t - b_{k_t})^{2s}}. \quad (5.4)$$

Note that by Lemma 3.3, the number of terms in the sum in (5.3) is bounded by

$$2^m \max\{|n-\alpha|^m, |n-\beta|^m\}. \quad (5.5)$$

Now, let $(j_1, k_1), \dots, (j_m, k_m) \in \mathcal{A}$, and let $t \in \{1, 2, \dots, m\}$. We consider the following possibilities (i), (ii) and (iii).

(i) If $a_{j_t} \in R_{j_t} \cap S_{k_t}$ and $b_{k_t} \in R_{j_t} \cap S_{k_t}$ we put $a'_{j_t} = a_{j_t}$ and $b'_{k_t} = b_{k_t}$. Note that if $a_{j_t} = b_{k_t}$, then both a_{j_t} and b_{k_t} belong to $R_{j_t} \cap S_{k_t}$.

(ii) If $a_{j_t} \in R_{j_t} \cap S_{k_t}$ but $b_{k_t} \notin R_{j_t} \cap S_{k_t}$, we put $a'_{j_t} = a_{j_t}$ and take b'_{k_t} to be the endpoint of $R_{j_t} \cap S_{k_t}$ that is closest to b_{k_t} . If $a_{j_t} \notin R_{j_t} \cap S_{k_t}$ but $b_{k_t} \in R_{j_t} \cap S_{k_t}$, we take a'_{j_t} to be the endpoint of $R_{j_t} \cap S_{k_t}$ that is closest to a_{j_t} , and we put $b'_{k_t} = b_{k_t}$.

(iii) If $a_{j_t} \notin R_{j_t} \cap S_{k_t}$ and $b_{k_t} \notin R_{j_t} \cap S_{k_t}$, it must happen that a_{j_t} lies to the left of $R_{j_t} \cap S_{k_t}$ and b_{k_t} to the right, or vice versa. In either case, we put a'_{j_t} to be one endpoint of $R_{j_t} \cap S_{k_t}$ and b'_{k_t} to be the other.

Now it is clear that in the cases (i) and (ii) above, for all $x_t \in R_{j_t} \cap S_{k_t}$ we have

$$|x_t - a_{j_t}|^{2s} |x_t - b_{k_t}|^{2s} \geq |x_t - a'_{j_t}|^{2s} |x_t - b'_{k_t}|^{2s}. \quad (5.6)$$

In case (iii) above, we see from the simple result on quadratics in Lemma 4.3 that (5.6) holds for all $x_t \in R_{j_t} \cap S_{k_t}$. All possibilities are exhausted by (i), (ii) and (iii), so that for all j_t, k_t we see that $a'_{j_t}, b'_{k_t} \in R_{j_t} \cap S_{k_t}$ and that (5.6) holds for all $x_t \in R_{j_t} \cap S_{k_t}$.

Now assume that $m \in \mathbb{N}$ with $m \geq 4s + 1$. We have from (5.4) and (5.6) that there is $M > 0$, depending on s and m only, such that

$$\begin{aligned} & J((j_1, k_1), \dots, (j_m, k_m)) \\ &= \int_{\prod_{t=1}^m R_{j_t} \cap S_{k_t}} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{t=1}^m |x_t - a_{j_t}|^{2s} |x_t - b_{k_t}|^{2s}} \\ &\leq \int_{\prod_{t=1}^m R_{j_t} \cap S_{k_t}} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{t=1}^m |x_t - a'_{j_t}|^{2s} |x_t - b'_{k_t}|^{2s}}, \text{ by (5.6),} \\ &\leq M \left(\max \{ \lambda(R_{j_1} \cap S_{k_1}), \lambda(R_{j_2} \cap S_{k_2}), \dots, \lambda(R_{j_m} \cap S_{k_m}) \} \right)^{m-4s}, \\ &\quad \text{by Lemma 4.2,} \\ &\leq \pi^{m-4s} M \min \left\{ \frac{1}{|n - \alpha|^{m-4s}}, \frac{1}{|n - \beta|^{m-4s}} \right\}, \text{ using (3.10),} \\ &= \frac{\pi^{m-4s} M}{\max\{|n - \alpha|^{m-4s}, |n - \beta|^{m-4s}\}}. \end{aligned} \quad (5.7)$$

Now, using (5.5), we see from (5.3) and (5.7) that

$$\begin{aligned} & \int_{[0, 2\pi]^m} \frac{dx_1 dx_2 \cdots dx_m}{\sum_{j=1}^m \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) x_j \right) \right|^{2s}} \\ &\leq \frac{2^{3m-6s} \pi^m M}{(n - \alpha)^{2s} (n - \beta)^{2s}} \cdot \frac{\max\{|n - \alpha|^m, |n - \beta|^m\}}{\max\{|n - \alpha|^{m-4s}, |n - \beta|^{m-4s}\}} \end{aligned}$$

$$\begin{aligned}
&= 2^{3m-6s} \pi^m M \cdot \max \left\{ \frac{(n-\alpha)^{2s}}{(n-\beta)^{2s}}, \frac{(n-\beta)^{2s}}{(n-\alpha)^{2s}} \right\} \\
&\leq 2^{3m-6s} \pi^m MK,
\end{aligned} \tag{5.8}$$

where $K > 0$ is a suitable constant chosen to be independent of n . Lemma 2.2 is immediate from (5.8) upon taking m to be $4s + 1$ and, as discussed in Section 2, Theorem 2.3 is now established. \square

6 A sharpness result

It was shown in Theorem 2.3 that if $f \in L^2([0, 2\pi])$ is such that $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$, then for almost all $(b_1, b_2, \dots, b_{4s+1}) \in [0, 2\pi]^{4s+1}$, f can be written in the form (2.6) and consequently in the form (2.5). However, in this section we show that if $m \in \mathbb{N}$ and $b_1, b_2, \dots, b_m \in [0, 2\pi]^{4s+1}$ are given, there are many functions with $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$ that cannot be written in the form (2.5). Thus, no single choice of b_1, b_2, \dots, b_m suffices to ensure that (2.5) is possible for all $f \in L^2([0, 2\pi])$ such that $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$. The methods extend the techniques in [5, pages 420-421].

Lemma 6.1 *Let $c_1, c_2, \dots, c_m \in \mathbb{R}$. Then, there are infinitely many $q \in \mathbb{N}$ such that $d_{\mathbb{Z}}(qc_j) < 1/q^{1/m}$ for all $j = 1, 2, \dots, m$.*

Proof. See [11, Theorem 4.6] or [13, page 27], for example. \square

Theorem 6.2 *Let $m, s \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{Z}$ be given. Also, let $c_1, c_2, \dots, c_m \in [0, 2\pi]$ be given. Then, there is a vector subspace V of $L^2([0, 2\pi])$ such that V has algebraic dimension equal to that of the continuum but, for any $f \in V$ with $f \neq 0$, there is no choice of $f_1, f_2, \dots, f_m \in L^2([0, 2\pi])$ such that f is equal to*

$$\sum_{j=1}^m \left[\left(e^{ic_j(\frac{\alpha-\beta}{2})} + e^{-ic_j(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left(e^{ic_j(\frac{\alpha+\beta}{2})} \delta_{c_j} + e^{-ic_j(\frac{\alpha+\beta}{2})} \delta_{-c_j} \right) \right]^s * f_j.$$

Proof. Let $f_1, f_2, \dots, f_m \in L^2([0, 2\pi])$, for $b \in [0, 2\pi]$ let λ_b be given by (2.1), and let f be given by

$$f = \sum_{j=1}^m \lambda_{c_j}^s * f_j. \tag{6.9}$$

Then, using (2.2), for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}\widehat{f}(n) &= \sum_{j=1}^m \widehat{\lambda}_{c_j}(n)^s \widehat{f}_j(n) \\ &= \sum_{j=1}^m \left(\cos \left(\left(\frac{\alpha - \beta}{2} \right) c_j \right) - \cos \left(\left(n - \frac{\alpha + \beta}{2} \right) c_j \right) \right)^s \widehat{f}_j(n) \\ &= 2^s \sum_{j=1}^m \sin^s \left(\frac{(n - \alpha)c_j}{2} \right) \sin^s \left(\frac{(n - \beta)c_j}{2} \right) \widehat{f}_j(n).\end{aligned}$$

Thus, as $|\sin x| \leq \pi d_{\mathbb{Z}}(x/\pi)$, we see that

$$|\widehat{f}(n + \alpha)| \leq 2^s \sum_{j=1}^m \left| \sin \left(\frac{nc_j}{2} \right) \right|^s |\widehat{f}_j(n + \alpha)| \leq \pi^s 2^s \sum_{j=1}^m d_{\mathbb{Z}} \left(\frac{nc_j}{2\pi} \right)^s |\widehat{f}_j(n + \alpha)|. \quad (6.10)$$

Now, by Lemma 6.1, for each $\ell \in \mathbb{N}$ and $k \in \{1, 2, \dots, 2^\ell\}$ there is $q_{k,\ell} \in \mathbb{Z}$ such that

$$d_{\mathbb{Z}} \left(q_{k,\ell} \left(\frac{c_j}{2\pi} \right) \right) < \frac{1}{\ell^{1/m}}, \text{ for all } j = 1, 2, \dots, m. \quad (6.11)$$

We see also from Lemma 6.1 that the integers $q_{k,\ell}$ can be chosen so that they are all distinct, over all $\ell \in \mathbb{N}$ and $k \in \{1, 2, \dots, 2^\ell\}$. Now, let Φ be the family of all functions $\phi : \mathbb{N} \mapsto \mathbb{N}$ such that $\phi(1) \in \{1, 2\}$ and $\phi(\ell + 1) \in \{2\phi(\ell) - 1, 2\phi(\ell)\}$ for all $\ell \in \mathbb{N}$. Note that if $\phi, \phi' \in \Phi$ and $\phi(j) \neq \phi'(j)$, then

$$\phi(n) \neq \phi'(n), \text{ for all } n > j. \quad (6.12)$$

We see that $\phi(n) \in \{1, 2, \dots, 2^n\}$ for all $n \in \mathbb{N}$, and that Φ has the cardinality of the continuum. Now, if $\phi \in \Phi$, define a function f_ϕ as follows. If $n \in \mathbb{Z}$ and $n = q_{\phi(\ell),\ell} + \alpha$ for some $\ell \in \mathbb{N}$, then ℓ is unique and we put

$$\widehat{f}_\phi(n) = \frac{1}{\ell^{1/2+s/m}}.$$

If $n \notin \{q_{\phi(\ell),\ell} + \alpha : \ell \in \mathbb{N}\}$, we put

$$\widehat{f}_\phi(n) = 0.$$

Then, because

$$\sum_{n=-\infty}^{\infty} |\widehat{f}_\phi(n)|^2 = \sum_{\ell=1}^{\infty} \frac{1}{\ell^{1+2s/m}} < \infty,$$

we see that $f_\phi \in L^2([0, 2\pi])$.

Now, assume that $\phi \in \Phi$ and that f_ϕ can be put in the form (6.9). By (6.10) and (6.11) and taking $n = q_{\phi(\ell), \ell}$ where $\ell \in \mathbb{N}$ we have

$$|\widehat{f}_\phi(q_{\phi(\ell), \ell} + \alpha)| \leq \frac{\pi^s 2^s}{\ell^{s/m}} \left(\sum_{j=1}^m |\widehat{f}_j(q_{\phi(\ell), \ell} + \alpha)| \right) \leq \frac{C \pi^s 2^s}{\ell^{s/m}} \left(\sum_{j=1}^m |\widehat{f}_j(q_{\phi(\ell), \ell} + \alpha)|^2 \right)^{1/2},$$

for some $C > 0$ that depends upon m only, and we see that

$$\sum_{\ell=1}^{\infty} \ell^{2s/m} |\widehat{f}_\phi(q_{\phi(\ell), \ell} + \alpha)|^2 < \infty. \quad (6.13)$$

However,

$$\sum_{\ell=1}^{\infty} \ell^{2s/m} |\widehat{f}_\phi(q_{\phi(\ell), \ell} + \alpha)|^2 = \sum_{\ell=1}^{\infty} \frac{\ell^{2s/m}}{\ell^{1+2s/m}} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = \infty. \quad (6.14)$$

The contradiction between (6.13) and (6.14) shows that f_ϕ cannot be put in the form (6.9).

Now, let $\phi_1, \phi_2, \dots, \phi_r \in \Phi$ be distinct, and consider a linear combination $h = \sum_{j=1}^r d_j f_{\phi_j}$ where, say, $d_1 \neq 0$. It follows from the observation (6.12) above that there is $k_0 \in \mathbb{N}$ such that for all $k > k_0$ with $\widehat{f}_{\phi_1}(k) \neq 0$, we have $\widehat{f}_{\phi_j}(k) = 0$ for all $j \in \{2, 3, \dots, r\}$. Thus, for all $k > k_0$ with $\widehat{f}_{\phi_1}(k) \neq 0$, we see that $\widehat{h}(k) = d_1 \widehat{f}_{\phi_1}(k)$. Then, if we apply (6.13) and (6.14) with f_{ϕ_1} in place of f_ϕ , we see that the contradiction between (6.13) and (6.14) applies to h , and we deduce that h cannot be written in the form (6.9). We now see that if V is the subspace of $L^2([0, 2\pi])$ finitely spanned by $\{f_\phi : \phi \in \Phi\}$, then V has the required properties. \square

7 Results for compact, connected abelian groups

In this section we look at some applications of the earlier results to compact, connected abelian groups and to the automatic continuity of linear forms on L^2 spaces on these groups.

Definitions. If $z \in \mathbb{T}$ and $\nu \in \mathbb{R}$ we may write z uniquely as $z = e^{it}$ where $t \in [0, 2\pi)$, and we then take z^ν to be $e^{it\nu}$. In particular, for $\alpha \in \mathbb{Z}$,

$z^{\alpha/2}$ is $e^{it\alpha/2}$. Let $\alpha, \beta \in \mathbb{Z}$ and let L be a linear form on $L^2(\mathbb{T})$. Then L is called (α, β) -invariant if, for all $b \in \mathbb{T}$ and $f \in L^2(\mathbb{T})$,

$$L \left[\left(b^{(\alpha+\beta)/2} \delta_b + b^{-(\alpha+\beta)/2} \delta_{b^{-1}} \right) * f \right] = (b^{(\alpha-\beta)/2} + b^{-(\alpha-\beta)/2}) L(f).$$

Also, L is called *translation invariant* if $L(\delta_b * f) = L(f)$ for all $b \in G$ and $f \in L^2(\mathbb{T})$.

Theorem 7.1 *Let $\alpha, \beta \in \mathbb{Z}$ and let L be a linear form on $L^2(\mathbb{T})$. Then the following hold.*

(i) *If L is (α, β) -invariant on $L^2(\mathbb{T})$, then L is continuous on $L^2(\mathbb{T})$. Thus, in this case there is a function $h \in L^2(\mathbb{T})$ such that $L(f) = \int_{\mathbb{T}} f \bar{h} d\mu_{\mathbb{T}}$ for all $f \in L^2(\mathbb{T})$.*

(ii) *If $h \in L^2(\mathbb{T})$, the linear form on $h \in L^2(\mathbb{T})$ given by $f \mapsto \int_{\mathbb{T}} f \bar{h} d\mu_{\mathbb{T}}$ is (α, β) -invariant if and only if there are $c_1, c_2 \in \mathbb{C}$ such that $h(z) = c_1 z^{\alpha} + c_2 z^{\beta}$ for almost all $z \in \mathbb{T}$.*

(iii) *If L is a translation invariant linear form on $L^2(\mathbb{T})$, L is a multiple of the Haar measure on \mathbb{T} .*

Proof. (i) It follows immediately from Theorem 2.3 with $s = 1$ that if L is (α, β) -invariant, L vanishes on the space

$$\{f : f \in L^2(\mathbb{T}) \text{ and } \widehat{f}(\alpha) = \widehat{f}(\beta) = 0\}.$$

Thus, if L is (α, β) -invariant it vanishes on this closed subspace of $L^2(\mathbb{T})$, a space that has finite codimension in $L^2(\mathbb{T})$. By (d) of Proposition 5.1 in [9, page 25], L is continuous. The existence of the function h in this case comes from the fact that the dual space of $L^2(\mathbb{T})$ is identified with $L^2(\mathbb{T})$.

(ii) If $h \in L^2(\mathbb{T})$, by Theorem 2.3 the linear form corresponding to h is (α, β) -invariant if and only if $\int_{\mathbb{T}} f \bar{h} d\mu_{\mathbb{T}} = 0$ whenever $\widehat{f}(\alpha) = \widehat{f}(\beta) = 0$. This occurs when $\widehat{h}(n) = 0$ for all $n \in \mathbb{Z}$ with $n \neq \alpha, \beta$. But that means that the Fourier expansion of h in $L^2(\mathbb{T})$ is a linear combination of z^{α} and z^{β} .

(iii) If L is translation invariant, we see that it is $(0, 0)$ invariant. Then, (ii) shows that L is a multiple of the Haar measure on \mathbb{T} . \square

Note that the conclusion (iii) in Theorem 7.1 is due to Meisters and Schmidt [5]. Conclusion (ii) generalises their result.

Let G denote a compact, connected abelian group with dual group \widehat{G} . The identity element in \widehat{G} is denoted by \widehat{e} . The group operation in such

a group will be written multiplicatively, and the normalised Haar measure on such a group G will be denoted by μ_G . We denote by $M(G)$ the family of bounded complex Borel measures on G . Let $m \in \mathbb{N}$ and for each $\gamma \in \widehat{G}$ with $\gamma \neq \widehat{e}$ let $h_\gamma : G^m \rightarrow \mathbb{T}^m$ be the function given by

$$h_\gamma(g_1, g_2, \dots, g_m) = (\gamma(g_1), \gamma(g_2), \dots, \gamma(g_m)).$$

The function h_γ is continuous and, as $\gamma \neq \widehat{e}$, h_γ maps G^m onto a compact connected subgroup of \mathbb{T}^m that is strictly larger than $\{1\}^m$, so this connected subgroup must be \mathbb{T}^m itself, as \mathbb{T}^m is connected. Consequently, h_γ maps G^m onto \mathbb{T}^m . It follows that for any non-negative measurable function on \mathbb{T}^m , we have

$$\int_{G^m} f \circ h_\gamma d\mu_{G^m} = \int_{\mathbb{T}^m} f d\mu_{\mathbb{T}}. \quad (7.1)$$

This is because each side of (7.1) defines a translation invariant integral over \mathbb{T}^m , so the equality in (7.1) is a consequence of the uniqueness of the Haar measure on \mathbb{T} , mentioned in (iii) of Theorem 7.1.

In the following result, note that for any compact connected abelian group G , for every element $b \in G$ we have $b = d^2$ for some $d \in G$ [3, vol. I, page 385].

Theorem 7.2 *Let G be a compact connected abelian group with dual group \widehat{G} . Let \widehat{e} be the identity element of \widehat{G} . Let $s \in \mathbb{N}$ and let $n, \alpha, \beta \in \mathbb{Z}$ with $n \notin \{\alpha, \beta\}$. Then, for a function $f \in L^2(G)$ the following conditions (i) and (ii) are equivalent.*

(i) *The Fourier transform of f vanishes at \widehat{e} . That is, $\widehat{f}(\widehat{e}) = 0$.*

(ii) *There are $b_1, b_2, \dots, b_{4s+1} \in G$ and $d_1, d_2, \dots, d_{4s+1} \in G$ with $d_j^2 = b_j$ for all $j \in \{1, 2, \dots, 4s+1\}$, such that there are $f_1, f_2, \dots, f_{4s+1} \in L^2(G)$ so that*

$$f = \sum_{j=1}^{4s+1} \left(\delta_{d_j^{\alpha-\beta}} + \delta_{d_j^{-(\alpha-\beta)}} - \delta_{d_j^{2n-(\alpha+\beta)}} - \delta_{d_j^{-(2n-(\alpha+\beta))}} \right)^s * f_j. \quad (7.2)$$

When f satisfies conditions (i) and (ii), almost all $(b_1, b_2, \dots, b_{4s+1}) \in G^{4s+1}$ have the following property: for any $d_1, d_2, \dots, d_{4s+1} \in G$ with $b_j = d_j^2$ for all $j \in \{1, 2, \dots, 4s+1\}$, there are $f_1, f_2, \dots, f_{4s+1} \in L^2(G)$ such that (7.2) holds.

In the case when $\widehat{f}(\widehat{e}) = 0$ and $\alpha - \beta$ is even, for almost all $(b_1, \dots, b_{4s+1}) \in G^{4s+1}$, there are $f_1, f_2, \dots, f_{4s+1} \in L^2(G)$ such that

$$f = \sum_{j=1}^{4s+1} \left(\delta_{b_j}^{(\alpha-\beta)/2} + \delta_{b_j}^{-(\alpha-\beta)/2} - \delta_{b_j}^{n-(\alpha+\beta)/2} - \delta_{b_j}^{-(n-(\alpha+\beta)/2)} \right)^s * f_j. \quad (7.3)$$

Proof. Assume that (i) holds. In (7.1), given $s \in \mathbb{N}$ and $n, \alpha, \beta \in \mathbb{Z}$ such that $n \neq \alpha$ and $n \neq \beta$, let's take f to be the function on \mathbb{T}^{4s+1} whose value $f(z_1, z_2, \dots, z_{4s+1})$ at $(z_1, z_2, \dots, z_{4s+1})$ is

$$\frac{1}{\sum_{j=1}^{4s+1} \left| z_j^{(\alpha-\beta)/2} + z_j^{-(\alpha-\beta)/2} - z_j^{n-(\alpha+\beta)/2} - z_j^{-(n-(\alpha+\beta)/2)} \right|^{2s}}.$$

Then, using (7.1) with $m = 4s + 1$, for each $\gamma \in \widehat{G}$ with $\gamma \neq \widehat{e}$ we have

$$\begin{aligned} & \int_{G^{4s+1}} \frac{d\mu_G(b_1) d\mu_G(b_2) \cdots d\mu_G(b_{4s+1})}{\sum_{j=1}^{4s+1} \left| \gamma(b_j)^{(\alpha-\beta)/2} + \gamma(b_j)^{-(\alpha-\beta)/2} - \gamma(b_j)^{n-(\alpha+\beta)/2} - \gamma(b_j)^{-(n-(\alpha+\beta)/2)} \right|^{2s}} \\ &= \frac{1}{2^{6s+1} \pi^{4s+1}} \int_{[0, 2\pi]^{4s+1}} \frac{dx_1 dx_2 \cdots dx_{4s+1}}{\sum_{j=1}^{4s+1} \left| \cos \left(\left(\frac{\alpha - \beta}{2} \right) x_j \right) - \cos \left(\left(n - \left(\frac{\alpha + \beta}{2} \right) \right) x_j \right) \right|^{2s}}. \end{aligned} \quad (7.4)$$

Now let $b, d \in G$ with $d^2 = b$, and let $\gamma \in \widehat{G}$. Then, $\gamma(d)^2 = \gamma(b)$, so that if we put $\gamma(b) = e^{i\theta}$ where $\theta \in [0, 2\pi)$, we have $\gamma(d) = e^{i\theta/2}$ or $\gamma(d) = e^{i(\theta/2+\pi)}$. In the former case we have

$$\gamma(d)^{\alpha-\beta} = e^{i\theta(\alpha-\beta)/2} = \gamma(b)^{(\alpha-\beta)/2}, \quad (7.5)$$

while in the latter case we have

$$\gamma(d)^{\alpha-\beta} = e^{i(\alpha-\beta)\pi} e^{i\theta(\alpha-\beta)/2} = (-1)^{\alpha-\beta} \gamma(b)^{(\alpha-\beta)/2}. \quad (7.6)$$

Similarly, when $\gamma(d) = e^{i\theta/2}$ or $\gamma(d) = e^{i(\theta/2+\pi)}$ we have, respectively,

$$\gamma(d)^{\alpha+\beta} = \gamma(b)^{(\alpha+\beta)/2} \text{ or } \gamma(d)^{\alpha+\beta} = (-1)^{\alpha+\beta} \gamma(b)^{(\alpha+\beta)/2}. \quad (7.7)$$

Note that in (7.5), (7.6) and (7.7), $\alpha - \beta$ and $\alpha + \beta$ are both even or both odd, so $(-1)^{\alpha-\beta}$ and $(-1)^{\alpha+\beta}$ are both equal to 1 or both equal to -1 .

Now, for $n \in \mathbb{Z}$ and $b, d \in G$ with $d^2 = b$, put

$$\lambda_{b,d,n} = \left(\delta_{d^{\alpha-\beta}} + \delta_{d^{-(\alpha-\beta)}} - \delta_{d^{2n-(\alpha+\beta)}} - \delta_{d^{-(2n-(\alpha+\beta))}} \right)^s \in M(G).$$

Then, for $\gamma \in \widehat{G}$,

$$\widehat{\lambda}_{b,d,n}(\gamma) = \left(\gamma(d)^{-(\alpha-\beta)} + \gamma(d)^{\alpha-\beta} - \gamma(d)^{-(2n-(\alpha+\beta))} - \gamma(d)^{2n-(\alpha+\beta)} \right)^s. \quad (7.8)$$

In view of (7.5), (7.6) and (7.7), we see that

$$|\widehat{\lambda}_{b,d,n}(\gamma)| = \left| \gamma(b)^{(\alpha-\beta)/2} + \gamma(b)^{-(\alpha-\beta)/2} - \gamma(b)^{n-(\alpha+\beta)/2} - \gamma(b)^{-(n-(\alpha+\beta)/2)} \right|. \quad (7.9)$$

Now, for each $b \in G$, let $d_b \in G$ be any element such that $d_b^2 = b$. As $n \notin \{\alpha, \beta\}$, if M is the constant as in Lemma 2.2 and we use (7.4) and (7.9), upon changing the order of summation and integration we have

$$\int_{G^{4s+1}} \left(\sum_{\gamma \in \widehat{G}, \gamma \neq \widehat{e}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^{4s+1} |\widehat{\lambda}_{b_j, d_{b_j}, n}(\gamma)|^2} \right) \prod_{j=1}^m d\mu_G(b_j) \leq \frac{M}{2^{6s+1} \pi^{4s+1}} \sum_{\gamma \in \widehat{G}} |\widehat{f}(\gamma)|^2, \quad (7.10)$$

which is finite by Plancherel's Theorem (see [3, vol. II, page 226]). We deduce that provided $\widehat{f}(\widehat{e}) = 0$, for almost all $(b_1, b_2, \dots, b_{4s+1}) \in G^{4s+1}$ we have that

$$\sum_{\gamma \in \widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^{4s+1} |\widehat{\lambda}_{b_j, d_{b_j}, n}(\gamma)|^2} < \infty. \quad (7.11)$$

Then, (ii) follows from (7.8), (7.11) and Theorem 2.1, so (i) implies (ii).

It is clear from (7.8) that if f has the form (7.2), then $\widehat{f}(\widehat{e}) = 0$. Thus, (ii) implies (i).

Above, the observation was made that (7.11) holds for almost all $(b_1, b_2, \dots, b_{4s+1}) \in G^{4s+1}$. If $(b_1, b_2, \dots, b_{4s+1})$ is any such point, we deduce that for any choice of elements $d_1, d_2, \dots, d_{4s+1}$ such that $d_j^2 = b_j$ for all j , and when $\widehat{f}(\widehat{e}) = 0$, we will have (7.2) holding.

When $\widehat{f}(\widehat{e}) = 0$ and $\alpha - \beta$ is even, the final conclusion derives from the above arguments and the fact that

$$d_j^{\alpha-\beta} = (d_j^2)^{(\alpha-\beta)/2} = b_j^{(\alpha-\beta)/2} \text{ and } d_j^{\alpha+\beta} = b_j^{(\alpha+\beta)/2}.$$

□

The following is a result concerning automatic continuity on groups. It is derived from Theorem 7.2, but only a special case is stated. More general results can be derived from Theorem 7.2.

Theorem 7.3 *Let G be a compact connected abelian group. Then the following conditions (i), (ii) and (iii) on a linear form $L : L^2(G) \rightarrow \mathbb{C}$ are equivalent.*

(i) *L is translation invariant. That is, $L(\delta_g * f) = L(f)$, for all $g \in G$ and $f \in L^2(G)$.*

(ii) *There is $n \in \mathbb{Z}$ with $n \notin \{-1, 1\}$ such that*

$$L((\delta_g + \delta_{g^{-1}}) * f) = L((\delta_{g^n} + \delta_{g^{-n}}) * f),$$

for all $g \in G$ and $f \in L^2(G)$.

(iii) *L is a multiple of the Haar measure.*

Also, the normalised Haar measure μ_G on G is unique.

Proof. (i) implies that (ii) holds for any $n \in \mathbb{N}$, so it must hold for any particular n . Also (iii) implies (i) because the Haar measure is translation invariant. Finally, assume that (ii) holds for some $n \in \mathbb{Z}$ with $n \notin \{-1, 1\}$. By Theorem 7.2 with $s = 1$, $\alpha = 1$ and $\beta = -1$, we deduce that L vanishes on the closed subspace $\{f : f \in L^2(G) \text{ and } \widehat{f}(\widehat{e}) = 0\}$. This latter space has codimension 1 in $L^2(G)$ and so it follows easily that L is continuous and is a multiple of the Haar measure on G (see [5, page 415]), and (iii) follows. Finally, as the Haar measure μ_G defines a translation invariant linear form on $L^2(G)$, the equivalence of (i) and (iii) implies the uniqueness of the Haar measure. □

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